

Lecture 8

02/11/2019

Review of Magnetostatics

Moving charges produce magnetic field. The Biot-Savart law describes the magnetic field due to a moving point charge q :

$$\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times (\vec{x} - \vec{x}_1)}{|\vec{x} - \vec{x}_1|^3} \quad (\mu_0 = \frac{1}{\epsilon_0 c^2})$$



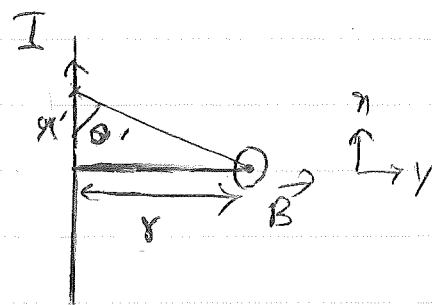
This relation is approximate and is valid at non-relativistic speeds. In general, for a current distribution $\vec{J}(\vec{x}_1)$, we have:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}_1) \times (\vec{x} - \vec{x}_1)}{|\vec{x} - \vec{x}_1|^3} d\tau_1$$

As an example, let us consider an infinite straight wire carrying a current I . In this case:

$$\vec{J}(\vec{x}_1) d\tau_1 = I d\ell' \hat{x}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\ell' \sin\theta'}{(r'^2 + r^2)^{3/2}} \hat{z} =$$



$$\frac{\mu_0 I}{2\pi r} \int_0^\pi \cos\theta' d\theta' \hat{z} = \frac{\mu_0 I}{2\pi r} \hat{z}$$

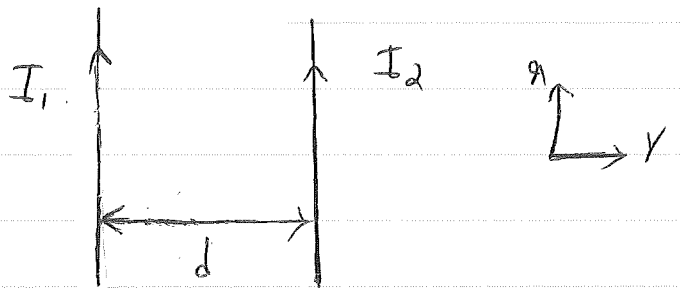
Magnetic fields exert force on moving charges. For a point charge q moving at velocity \vec{v} , the Lorentz force is:

$$\vec{F} = q\vec{v} \times \vec{B}$$

For a general current distribution, the total force is:

$$\vec{F} = \int \vec{J}(\vec{x}') \times \vec{B}(\vec{x}') d\tau'$$

For example, consider two parallel infinite wires carrying currents I_1 and I_2 , respectively, shown here:



The force per unit length of wire 2

is given by:

$$\vec{F}_2 = -I_2 \int_0^l dl_2 B_1(\vec{x}_2) \hat{y} = \frac{-\mu_0 I_1 I_2}{2\pi d} \hat{y}$$

We note that $B_2(\vec{x}_2) = 0$ anywhere on wire 2. Similarly, it can be

shown that:

$$\vec{F}_1 = I_1 \int_0^l dl_1 B_2(\vec{x}_1) \hat{y} = \frac{\mu_0 I_1 I_2}{2\pi d} \hat{y}$$

As mentioned, magnetic field due to a current distribution \vec{J} is:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d\tau'$$

However:

$$\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \Rightarrow \vec{B}(\vec{x}) = \frac{-\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d\tau'$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d\tau'$$

Here, we have used:

$$\vec{\nabla} \times \left(\vec{J}(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|} \right) = -\vec{J}(\vec{x}') \times \vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) + \frac{1}{|\vec{x} - \vec{x}'|} \vec{\nabla} \times \vec{J}(\vec{x}')$$

(since $\vec{\nabla}$ does not operate on \vec{x}')

Therefore, we can write:

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d\tau'$$

An important consequence of $\vec{B} = \vec{\nabla} \times \vec{A}$ is that:

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \text{no magnetic monopoles}$$

The quantity $\vec{A}(\vec{x})$ is called the vector potential. However,

we note that $\vec{A}(\vec{x})$ is not unique. In fact, if $\vec{B} = \vec{\nabla} \times \vec{A}$, then

$\vec{A}' = \vec{A} + \vec{\nabla} \chi$ (where χ is an arbitrary scalar function) will give exactly the same \vec{B} . This is called a gauge transformation and implies that \vec{A} is not a physically observable quantity.

There are a couple of important properties for the current density $\vec{J}(\vec{x})$ in magnetostatics. First, conservation of the electric charge implies that:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Since there is no explicit time dependence in the case of magnetostatics,

$$\frac{\partial \rho}{\partial t} = 0, \text{ and hence:}$$

$$\vec{\nabla} \cdot \vec{J} = 0$$

Second, consider a localized charge distribution for which $J(\vec{x})$ drops quickly as $r = |\vec{x}|$ grows. In this case:

$$\int_V \partial_i (\epsilon_0 J_i) d\tau = \int_V \vec{\nabla} \cdot \vec{K} d\tau = \oint_S \vec{K} \cdot \hat{n} da = \oint_S \epsilon_0 J_i n_i da$$

Einstein summation convention implied $K_i \equiv \epsilon_0 J_i$

Taking the boundary S to infinity, since $\vec{J}(\vec{x})$ is localized, results in the following:

$$\int_V \partial_i (n_j J_j) d\tau = 0$$

However:

$$\int_V \partial_i (n_j J_j) d\tau = \int_V (\partial_i n_j) J_j d\tau + \int_V n_j (\partial_i J_j) d\tau \Rightarrow \overset{\vec{\nabla} \cdot \vec{J} = 0}{\Rightarrow}$$

$$\int_V (\partial_i n_j) J_j d\tau = \int_V \delta_{ij} J_j d\tau = \int_V J_i d\tau = 0 \Rightarrow \boxed{\int_V \vec{J} d\tau = 0}$$

Writing $\vec{B} = \vec{\nabla} \times \vec{A}$, we find:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

But:

$$\vec{\nabla} \cdot \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \cdot \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d\tau' = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \cdot \vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d\tau'$$

$$= -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d\tau'$$

However:

$$\vec{J}(\vec{x}_1) \cdot \vec{\nabla}' \left(\frac{1}{|\vec{x} - \vec{x}_1|} \right) = \vec{\nabla}' \cdot \left(\frac{\vec{J}(\vec{x}_1)}{|\vec{x} - \vec{x}_1|} \right) - \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}_1)}{|\vec{x} - \vec{x}_1|}$$

Therefore:

$$\vec{\nabla} \cdot \vec{A} = -\frac{\mu_0}{4\pi} \int \vec{\nabla}' \cdot \left(\frac{\vec{J}(\vec{x}_1)}{|\vec{x} - \vec{x}_1|} \right) d\sigma_1 \stackrel{\rightarrow}{=} 0 \quad \text{for a localized current distribution (using the divergence theorem)}$$

This results in:

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= -\nabla^2 \vec{A} = -\frac{\mu_0}{4\pi} \nabla^2 \int \frac{\vec{J}(\vec{x}_1)}{|\vec{x} - \vec{x}_1|} d\sigma_1 = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}_1) \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}_1|} \right) d\sigma_1 \\ &= \mu_0 \int \vec{J}(\vec{x}_1) \delta^{(3)}(\vec{x} - \vec{x}_1) d\sigma_1 \end{aligned}$$

This gives rise to the Ampere's law for magnetostatics:

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

Magnetic Fields of a Localized Current Distribution

The vector potential due to a localized steady current distribution

$\vec{J}(\vec{x})$ is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}_1)}{|\vec{x} - \vec{x}_1|} d\sigma_1 = \frac{\mu_0}{4\pi} \int \left[\frac{1}{r} + \frac{\vec{x} \cdot \vec{x}_1}{r^3} + O\left(\frac{r_1^2}{r^3}\right) \right] \vec{J}(\vec{x}_1) d\sigma_1$$

However:

$$\frac{\mu_0}{4\pi} \int \frac{1}{r} \vec{J}(\vec{x}') d\tau' = \frac{\mu_0}{4\pi r} \int \vec{J}(\vec{x}') d\tau' = 0$$

vanishes for a localized distribution

This is just a manifestation that there are no magnetic monopoles.

Therefore, the first non-vanishing term is:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{\vec{x}}{r^3} \cdot \int \vec{x}' \vec{J}(\vec{x}') d\tau' + \dots$$

We can write:

$$\vec{x} \cdot \int \vec{x}' \vec{J}(\vec{x}') d\tau' = \hat{e}_i x_j \int x'_j J_i(\vec{x}') d\tau' \quad (\hat{e}_i; \text{unit vector along the } i\text{-th direction})$$

$$x'_j J'_i(\vec{x}') = \frac{1}{2} \underbrace{(x'_j J'_i + x'_i J'_j)}_{\text{symmetric under } i \leftrightarrow j} + \frac{1}{2} \underbrace{(x'_j J'_i - x'_i J'_j)}_{\text{anti-symmetric under } i \leftrightarrow j}$$

Now:

$$\int (x'_j J'_i + x'_i J'_j) d\tau' = \int \left[x'_j \frac{\partial}{\partial x'_k} (x'_i J'_k) + x'_i J'_j \right] d\tau'$$

Here, we have used:

$$\frac{\partial}{\partial x'_k} (x'_i J'_k) = \delta_{ik} J'_k + x'_i \underbrace{\frac{\partial}{\partial x'_k} J'_k}_{\vec{\nabla} \cdot \vec{J} = 0} = J'_i$$

Note that:

$$q'_j \frac{\partial}{\partial q'_k} (q'_i J_k) = \frac{\partial}{\partial q'_k} (q'_j q'_i J_k) - \left(\frac{\partial q'_j}{\partial q'_k} \right) q'_i J_k$$

Thus:

$$\int \left[q'_j \frac{\partial}{\partial q'_k} (q'_i J_k) + q'_i J_j \right] d\sigma' = \oint_S q'_j q'_i \vec{J} \cdot \hat{n}' da' - \int q'_i J_j d\sigma' + \int q'_i J_j d\sigma'$$

vanishes for a localized distribution

As a result:

$$\vec{x} \cdot \int \vec{x}' \vec{J}(\vec{x}') d\sigma' = \frac{1}{2} \hat{e}_j q_j \int [q'_j J_i(\vec{x}') - q'_i J_j(\vec{x}')] d\sigma'$$

Hence:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r^3} \vec{x} \times \left(\frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}') d\sigma' \right) + \dots \Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi r^3} \vec{m} \times \vec{x} + \dots$$

Here, \vec{m} is the magnetic dipole moment of the distribution:

$$\vec{m} \equiv \frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}') d\sigma'$$

The magnetic moment density is defined as:

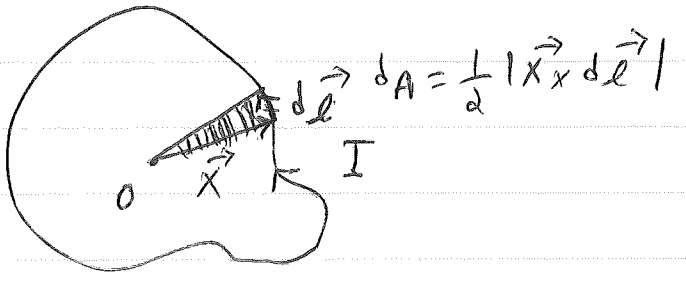
$$\vec{j}(\vec{x}) = \frac{1}{2} \vec{x} \times \vec{J}(\vec{x})$$

For a current loop carrying a current I , we have:

$$\vec{m} = \frac{I}{2} \oint_C \vec{x} \times d\vec{\ell}$$

For a planar loop, $\vec{x} \times d\vec{\ell}$ is in the direction of the normal to the loop, \hat{n} , and its magnitude is dA :

$$\vec{m} = I \int dA \hat{n} = IA\hat{n}$$



For a point magnetic dipole at the origin, the expression $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{r^3}$

is exact. Then:

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \nabla \times \left(\frac{\vec{m} \times \vec{x}}{r^3} \right) = -\frac{\mu_0}{4\pi} \nabla \times \left(\vec{m} \times \nabla \left(\frac{1}{r} \right) \right) = -\frac{\mu_0}{4\pi} \left[(\nabla \cdot \nabla \left(\frac{1}{r} \right)) \vec{m} - (\vec{m} \cdot \nabla) \nabla \left(\frac{1}{r} \right) \right]$$

We note that $\nabla^2 \left(\frac{1}{r} \right) = 4\pi \delta^{(3)}(\vec{x})$. Therefore, at any point other than the origin, we have:

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} (\vec{m} \cdot \nabla) \nabla \left(\frac{1}{r} \right) = -\frac{\mu_0}{4\pi} (\vec{m} \cdot \nabla) \left(\frac{\vec{x}}{r^3} \right) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \vec{x})\vec{x}}{r^5} \right.$$

$$\left. - \frac{\vec{m}}{r^3} \right] = \frac{\mu_0}{4\pi} \left[\frac{3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m}}{r^3} \right] \quad (\hat{n} \equiv \frac{\vec{x}}{r})$$